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Research on Hawking Radiation as Tunneling from Schwarzshild-anti-de Sitter Black Hole

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After taking into account energy conservation and the particle's self-gravitation interaction, Hawking radiation of the massive particle as tunneling from Schwarzshild-anti-de Sitter black hole is studied by using Parikh-Wilczek's semi-classical quantum tunneling approach. Meanwhile, Hawking radiation as tunneling from the black hole is reexamined by developing Angheben–Nadalini–Vanzo–Zerbini (ANVZ) covariant method to cover energy conservation and the particle's self-gravitation interaction. Both the results perfectly generalize those obtained by Parikh and Wilczek, and show that the tunneling rate is related to the change of Bekenstein-Hawking entropy, and the factual emission spectrum is not exactly thermal, but satisfies the underlying unitary theory.

KEY WORDS: Schwarzshild-anti-de Sitter black hole; tunneling rate; energy conservation; self-gravitation.

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1. INTRODUCTION

In1974, Hawking proved that black hole can radiate thermally, and the temperature is the true quantum temperature (Hawking, 1975). The created mechanism of Hawking radiation can be explained by the tunneling effect in Quantum Mechanism or the vacuum fluctuation in Quantum Field Theory (namely, a pair of particles created just inside the horizon, the negative energy particle is absorbed by black hole, and the positive is escaped infinity to form Hawking radiation. In other words, we can also consider that the particles created just outside the horizon, the negative energy particle is tunneling into the horizon, and the positive is left outside the horizon and moves towards the infinite distance and forms Hawking thermal spectrum). Both depictions have a tunneling process, but actual derivation of Hawking radiation did not proceed in this way at all, and are most based on the

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quantum field theory on a fixed background space-time without considering the fluctuation of the space-time geometry (Zhao and Zhu, 1999; Liu and Li, 1999; Jing, 2001; Li and Mi, 1999; Wu and Cai, 2002; Li, 2005; Jiang *et al.*, 2005). So the urgency of finding the theory to derive Hawking radiation spectrum based on the dynamical space-time is necessary.

Recently, Parikh and Wilczek have adopted the semi-classical quantum tunneling picture to study the tunneling radiation of Schwarzshild and Reissner-Nordstrom black holes (Parikh and Wilczek, 2000), the tunneling rate is obtained by considering energy conservation and the particle's self- gravitation interaction. The result shows that the exact radiation spectrum is no longer pure thermal after considering the black hole background as dynamical. Since then, a lot of work has already been carried out for further development of the approach, and all the obtained results are very successful to support Parikh-Wilzcek's prescription (Zhang and Zhao, 2005a,b,c; Liu, 2006; Wu and Jiang, 2006; Jiang and Wu, 2006). Nevertheless, all these investigations are not completely satisfactory because they are only restricted to the metrics to constant-time slices in flat Euclidean in radial or in de Sitter space in radial. In the approach, the Painlevé coordinate system introduced to eliminate the coordinate singularity has many superior characters, e.g. The metric is regular at the event horizon; There exists a time-like vector throughout the Painlevé coordinate system; the metric after undergoing the coordinate transformation satisfies Landau's condition of coordinate clock synchronization. Although the Painlevé coordinate system has so many superior characters that it carries us convenience to study the tunneling effect of Hawking radiation, it is difficult to find a unified Painlevé coordinate system. Moreover, The geodesics for the massless particle tunneling across the event horizon is light-like, however, that of the massive particle is not, but difficultly decided by the de Broglie's hypothesis and the definition of the phase (group) velocity. Finally, the integration over the emitted particle's energy ω is necessary but cumbersome for the approach.

Fortunately, Angheben, Nadalini, Vanzo and Zerbini (ANVZ) have recently developed a new method to revisit Hawking radiation as tunneling (Angheben *et al.*, 2005). All the above difficulties existed in Parikh-Wilczek's semi-classical quantum tunneling method all at once are overcome during doing explicit computation on Hawking tunneling radiation. And the novelty of the approach has mainly been consisted in the covariant treatment of the horizon singularity through the use of spatial proper distance. As a result, the correct Hawking temperature in the static Schwarzshild gauge has been derived.

In the paper, we will first introduce Parikh-Wilczek's semi-classical quantum tunneling approach to discuss Hawking radiation of the massive particle via tunneling from Schwarzshild-anti-de Sitter black hole when the particle's self-gravitation interaction is taken into account, and then, we will develop Angheben–Nadalini–Vanzo–Zerbini covariant approach to cover the effect of back reaction to further revisit Hawking radiation as tunneling. The results show that the tunneling rate is related to the change of Bekenstein-Hawking entropy and the factual emission spectrum deviates from the pure thermal spectrum, but is consistent with an underlying unitary theory.

2. HAWKING RADIATION OF THE MASSIVE PARTICLE VIA TUNNELING

The four-dimension Schwarzshild-Anti-de Sitter metric can be written as

$$ds^2 = -f(r)\, dt_S^2 + \frac{1}{f(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1}$$

where $f(r) = 1 - \frac{2M}{r} + \frac{r^2}{R^2}$, and $\Lambda = -\frac{3}{R^2}$ is a negative cosmological constant.

There exists a coordinate singularity in the metric (1) at the event horizon, so performing the Painlevé coordinate transformation via (Painleve, 1921)

$$dt_S = dt - \frac{1}{f(r)}\sqrt{1 - \frac{R^2}{R^2 + r^2} f(r)}\, dr, \tag{2}$$

yields

$$ds^2 = -f(r)\, dt^2 + 2\sqrt{1 - \frac{R^2}{R^2 + r^2} f(r)}\, dt dr$$
$$+ \frac{R^2}{R^2 + r^2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{3}$$

where the new line element provides many attractive features to study the tunneling behavior across the event horizon: (1) the metric is well-behaved at the event horizon; (2) ∂_t is still a Killing vector through the whole new space-time; (3) the infinite red-shift coincides with the event horizon; (4) it satisfies Landau's condition of the coordinate clock synchronization. These attractive features are very advantageous to do an explicit computation of the tunneling probability at the event horizon.

In the subsequent part, we shall discuss the tunneling behavior of the massive particle. Now, let's work with the new metric (3) and obtain the radial geodesics of the massive particle. According to Zhang and Zhao (2005a,b), the trajectory followed by massive particle is not light-like, but decided by de Broglie's hypothesis and the definition of the phase (group) velocity, and can be expressed as

$$\dot{r} = v_p = \frac{1}{2} v_g = -\frac{g_{tt}}{2g_{tr}} = \frac{f(r)}{2\sqrt{1 - \frac{R^2}{R^2+r^2} f(r)}} \approx \kappa_h(M, r_h)(r - r_h). \tag{4}$$

After considering the particle's self-gravitation interaction, the mass parameters in Eqs. (3) and (4) will be replaced with $M \to M - \omega$ when the particle of energy ω tunnels out of the event horizon. The action of the massive particle across the event horizon can be expressed as

$$S = \int_{t_i}^{t_f} L dt = \int_{r_i}^{r_f} \int_0^{P_r} dP_r' dr, \tag{5}$$

where r_i and r_i are the locations of the event horizon before and after the particle of energy ω emission. Introducing the Hamilton's equation as follows

$$\dot{r} = \left. \frac{dH}{dP_r} \right|_r = \frac{d\,(M-\omega)}{dP_r}, \tag{6}$$

therefore, the imaginary part of the action is

$$\begin{aligned} \mathrm{Im} S &= \mathrm{Im} \int_{r_i}^{r_f} \int_M^{M-\omega} \frac{dr}{\dot{r}} d(M-\omega') \\ &= \mathrm{Im} \int_{r_i}^{r_f} \int_M^{M-\omega} \frac{2\sqrt{1 - \frac{R^2}{R^2+r^2} f'(r)}}{f'(r)} dr d(M-\omega') \\ &= -\int_{(M,Q)}^{(M-\omega,Q-q)} \frac{\pi}{\kappa_h(M-\omega', r_h')} d(M-\omega') = -\frac{1}{2} \Delta S_{BH}, \end{aligned} \tag{7}$$

where we use the first law of thermodynamics of the black hole as

$$dS_{BH} = \frac{dE'}{T'} = \frac{d(M-\omega')}{T'}. \tag{8}$$

So the tunneling rate can be expressed as

$$\Gamma \sim e^{-2\,\mathrm{Im\,S}} = e^{\Delta S_{BH}}, \tag{9}$$

where $\Delta S_{BH} = S_{BH}(M-\omega) - S_{BH}(M)$ is the change of Bekenstein-Hawking entropy before and after the particle emission. Here, we can easily see that the tunneling rate is related to the difference of the entropy before and after the massive particle emission, which perfectly generalizes those obtained by Parikh and Wilczek and is indicative of a consistence with an underling unitary theory. In the next, we will introduce a unified covariant framework, first initiated by Angheben, Nadalini, Vanzo and Zerbini, for revisiting Hawking radiation behavior across the event horizon of the black hole.

3. ANVZ COVARIANT APPROACH AND HAWKING TUNNELING RADIATION

The classical action S of the emitted particle tunneling across the horizon satisfies the relativistic Hamilton-Jacobi equation as

$$g^{\mu\nu}\partial_\mu S\partial_\nu S + u^2 = 0, \tag{10}$$

where u is the mass of the emitted particle. Substituting the inverse metric derived from the line element (1) into Eq. (10) yields

$$-\frac{1}{f(r)}(\partial_t S)^2 + f(r)(\partial_r S)^2 + \frac{1}{r^2}(\partial_\theta S)^2 + \frac{1}{r^2\sin^2\theta}(\partial_\varphi S) + u^2 = 0, \tag{11}$$

Due to the symmetry of the space-time, the action can be written as

$$S = -\omega t + P_\varphi\varphi + P_\theta\theta + W(r), \tag{12}$$

where ω is the energy of the emitted particle and (P_φ, P_φ) are the angular momentum with respect to the angular (φ, θ). Substituting Eq. (12) into Eq. (11) can we obtain

$$\frac{\partial W(r)}{\partial r} = \frac{1}{f(r)}\sqrt{\omega^2 - f(r)\left(\frac{1}{r^2}P_\theta^2 + \frac{1}{r^2\sin^2\theta}P_\varphi^2 + u^2\right)}. \tag{13}$$

Direction evaluation of the pole contribution is highly coordinate dependent because of lack of covariance, and so making use of the proper special distance first observed by ANVZ, namely

$$d\sigma^2 = \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{14}$$

and limiting to the s-wave contribution that is contained the bulk of particle emission yields

$$\begin{aligned}\sigma &= \int\frac{dr}{\sqrt{f(r)}} = \frac{2}{\sqrt{f_{,r}(r_h)}}\sqrt{r - r_h} + \cdots\cdots \\ &= \sqrt{\frac{2}{\kappa_h(M, r_h)}}\sqrt{r - r_h} + \cdots\cdots, \frac{1}{\sqrt{f(r)}} \\ &= \frac{2}{f_{,r}(r_h)}\frac{1}{\sigma} + \cdots\cdots = \frac{1}{\kappa_h(M, r_h)}\frac{1}{\sigma} + \cdots\cdots.\end{aligned} \tag{15}$$

Substituting Eqs. (14) and (15) into Eq. (13), we have

$$W(\sigma) = \frac{1}{\kappa_h(M, r_h)}\int\frac{d\sigma}{\sigma}\sqrt{\omega^2 - f(r)\left(\frac{1}{r^2}P_\theta^2 + \frac{1}{r^2\sin^2\theta}P_\varphi^2 + u^2\right)}. \tag{16}$$

Deforming the integration contour from the real σ-axis to the lower complex σ-plane which avoids the pole $\sigma = 0$ counterclockwise, and using the Feynman prescription at the event horizon as

$$\frac{1}{\sigma} \to \frac{1}{\sigma - i0} = PP\frac{1}{\sigma} + i\pi\delta(\sigma), \tag{17}$$

we can obtain the action across the event horizon as

$$S = \frac{\pi i}{\kappa_h(M, r_h)}\omega + \text{(real contribution)}. \tag{18}$$

Now, let's incorporate the self-gravitation interaction during the emitted particle tunneling across the horizon of black hole, and move on to discuss Hawking radiation as tunneling. Considering the emitted particle as a shell of energy ω', and fixing the total mass of the space-time and allowing that of the black hole to fluctuate, when a particle tunnels out, the black hole's mass will become $M - \omega'$. So if the self-gravitation interaction is taken into account, the corrected action should be written as

$$\operatorname{Im} S_c = \operatorname{Im} \int dS|_{(M \to M - \omega')} = \int_0^{\omega} \frac{\pi}{\kappa_h(M - \omega', r_h')} d\omega', \tag{19}$$

Rewrite Eq. (30) in a more concise form as

$$\operatorname{Im} S_c = -\int_M^{M-\omega} \frac{\pi}{\kappa_h(M - \omega', r_h')} d(M - \omega') = -\frac{1}{2}\Delta S_{BH}, \tag{20}$$

where the first law of the thermodynamics of the black hole is used. So the tunneling probability is

$$\Gamma \sim e^{-2\,\mathrm{Im}\,\mathrm{S_c}} = e^{\Delta S_{BH}}, \tag{21}$$

where $\Delta S_{BH} = S(M - \omega) - S(M)$ is the change of Bekenstein-Hawking entropy before and after the particle emission. So the quantum-corrected tunneling rate as a result of the self-gravitation interaction is connected with the change of Bekenstein-Hawking entropy, and the factual spectrum of the black hole deviates from the pure thermal spectrum, but is consistent with an underlying unitary theory.

4. DISCUSSION AND CONCLUSION

In the paper, we have visited Hawking radiation of the massive particle as a semiclassical tunneling process from the event horizon of Schwarzshild-anti-de Sitter black hole by the particle's self-gravitation interaction. The pertinent points of this approach are that Hawking radiation is a dynamical mechanism and the connected conservation laws must be incorporated when doing explicit computation for the tunneling probability across the horizon; Moreover, the Painlevé coordinate transformation is introduced to eliminate the coordinate singularity and provides a clear tunneling process across the horizon; Thirdly, de Broglie's hypothesis and the definition of the phase (group) velocity is applied to obtain the geodesics of the massive particle. The derived results show that after considering the particle's self-gravitation interaction, the tunneling rate of the massive particle is related to the change of Bekenstein-Hawking entropy and the factual radiation spectrum deviates from the pure thermal spectrum, but is consistent with the underlying unitary theory.

For another, considering the effect of the particle's self-gravitation interaction, we have adopted and developed Angheben–Nadalini–Vanzo–Zerbini (ANVZ) covariant approach to further revisit Hawking radiation as tunneling from Schwarzshild-anti-de Sitter black hole. And the novelty of the approach has mainly been consisted in the covariant treatment of the horizon singularity through the use of spatial proper distance. As a result, we can also get the same result perfectly generalize those obtained by Parikh and Wilczek.

In summary, both the derived results support Parikh-Wilczek's opinion, and show that the tunneling rate is proportional to the exponential of the change of Bekenstein–Hawking entropy, which is indicative of a consistence with an underling unitary theory, and the factual radiation spectrum is not exactly thermal but has subtle corrections.

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